**Trilateral Double Generating Formulae for Polynomials**

Agarwal P1*, Nisar K.S2, Khan M.A3 and Jain S4

1Department of Mathematics, Anand International College of Engineering, India
2Department of Mathematics, Prince Sattam bin Abdulaziz University, Saudi Arabia
3Department of Applied Mathematics, Aligarh Muslim University, India
4Department of Mathematics, Poornima College of Engineering, India

**Abstract**

In this paper, we have obtained a new class of trilateral double generating function containing Legendre, Laguerre, Hermite, and Jacobi polynomials in terms of operational representations.

**INTRODUCTION**

Orthogonal polynomials constitute an important class of special functions in general and hypergeometric functions in particular. The subject of orthogonal polynomials is a classical one whose origins can be traced to Legendre’s work on planetary motion with important applications to physics and to probability and statistics and other branches of mathematics, the subject flourished through the first third of the century. Perhaps as a secondary effect of the computer revolution and the heightened activity in approximation theory and numerical analysis, interest in orthogonal polynomials has revived in recent years. During last four decades or so, a number of authors have established various families of generating functions of polynomials in many different ways [1-3].

In [4] Khan and Nisar gave partial differential operator representations of bilinear and bilateral summation formulae for various polynomials whose Rodrigues or Rodrigues type formula are known. In this paper, we derive trilateral generating function of various polynomials. For deriving the trilateral double generating function containing Legendre, Laguerre, Hermite, and Jacobi polynomials we recall the results used by Khan and Shukla in [5].

\[ D^\mu \lambda = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda - \mu)} x^{\lambda-\mu}, \quad D = \frac{d}{dx} \]  

(1.1)

where \( \lambda \) and \( \mu, \lambda \geq \mu \) are arbitrary real numbers.

In particular, use has been made of the following results:

\[ D^{r} e^{-x} = (-1)^r e^{-x} \]  

(1.2)

\[ D^{r} x^{-\alpha} = (\alpha)_r (-1)^r x^{-\alpha-r}, \alpha \text{ is not an integer} \]  

(1.3)

\[ D^{r} x^{-n-\alpha} = (\alpha + n)_r (-1)^r x^{-\alpha-n-r} \]  

(1.4)

\[ D^{n-r} x^{-\alpha} = \frac{(\alpha)_n}{(\alpha)_r} x^{\alpha-1+r} \]  

(1.5)

\[ D^{n-r} x^{-\alpha} = \frac{(\alpha)_n(-1)^n}{(1-\alpha-n)_r} x^{-\alpha+n+r}, \alpha \text{ is not an integer} \]  

(1.6)

Where \( n \) and \( r \) are denote positive integers and

\[ (\alpha)_n = \alpha (\alpha + 1) \cdots (\alpha + n - 1); \quad (\alpha)_0 = 1 \]

\[ D^{n-r} \left\{ x^{\alpha+n} e^{-x} \right\} = (n-r)! x^{\alpha+r} e^{-x} \binom{\alpha+r}{n-r}(x) \]  

(1.7)
We also need the definition of the following polynomials [6-9]

### Legendre Polynomial

It is denoted by the symbol $P_n(x)$ and is defined as

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{(n-k)!}{(n+k)!} x^{n+k}$$

### Laguerre Polynomials

It is denoted by the symbol $L_n^{(\alpha)}(x)$ and is defined as

$$L_n^{(\alpha)}(x) = \frac{1}{\alpha^n n!} \left[ x^{\alpha} \right]^{n}$$

### Jacobi Polynomial

It is denoted by the symbol $P_n^{(\alpha, \beta)}(x)$ and is defined as

$$P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)_n}{n!} \left[ x^{1+\alpha} \right]^{n}$$

### Ultraspherical Polynomial

The special case of $\beta = \alpha$ of the Jacobi polynomials is called ultraspherical polynomial and is denoted by $P_n^{(\alpha, \alpha)}(x)$. It is defined as

$$P_n^{(\alpha, \alpha)}(x) = \frac{(1 + \alpha)_n}{n!} \left[ x^{1+\alpha} \right]^{n}$$

### Hermite Polynomial

It is denoted by the symbol $H_n(x)$ and is defined as

$$H_n(x) = (2x)^n e^{-x^2}$$
\[ D^n(uv) = \sum_{r=0}^{n} nC_r D^{n-r}u D^r v \]  
(2.1)

\[ D^n(uvw) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1)^{r+s}}{r!s!} D^{n-r-s}u D^r v D^s w \]  
(2.2)

If \( D_x \equiv \frac{\partial}{\partial x} \) and \( D_y \equiv \frac{\partial}{\partial y} \), \( D_z \equiv \frac{\partial}{\partial z} \)
Khan and Shukla [5] wrote the binomial expansion for \( (D_x + D_y + D_z)^n \) as

\[ (D_x + D_y + D_z)^n = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1)^{r+s}}{r!s!} D_x^{n-r-s}D_y^r D_z^s \]  
(2.3)

operating (2.2) on \( F(x,y,z) \), we get

\[ (D_x + D_y + D_z)^n F(x,y,z) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1)^{r+s}}{r!s!} D_x^{n-r-s}D_y^r D_z^s F(x,y,z) \]  
(2.4)

In particular, if \( F(x,y,z) = f(x)g(y)h(z) \), then (2.4) gives

\[ (D_x + D_y + D_z)^n \{ f(x)g(y)h(z) \} = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1)^{r+s}}{r!s!} D_x^{n-r-s}D_y^r D_z^s f(x)D_y^r g(y)D_z^s h(z) \]  
(2.5)

In [4] Khan and Nisar gave partial differential operator representations of bilinear and bilateral summation formulae for various polynomials whose Rodrigues or Rodrigues type formula are known and they gave the following results:

\[ (D_x + D_y)^n \{ x^{\alpha+n} y^{\beta+n} e^{-x-y} \} = n! x^\alpha y^{\beta+n} e^{-x-y} \sum_{t=0}^{n} \binom{\alpha+r}{t} \binom{\beta+n+r}{t} \left( \frac{x}{y} \right)^t \]  
(2.6)

\[ (D_x + D_y)^n \{ e^{-x^2 - y^2} \} = (-1)^n e^{-x^2 - y^2} \sum_{t=0}^{n} \frac{n!}{r! (n-r)!} H_{n-r}(x) H_r(y) \]  
(2.7)

\[ (D_x + D_y)^n \{ (x-1)^{\alpha+n} (x+1)^{\beta+n} (y-1)^{\gamma+n} (y+1)^{\delta+n} \} = 2^n n! (x-1)^\alpha (x+1)^\beta (y-1)^\gamma (y+1)^\delta \]  
(2.8)

\[ \sum_{t=0}^{n} p_{\alpha+n,\beta+r}(x)p_{\gamma+n,\delta-r}(y) \left( \frac{x^2-1}{y^2-1} \right)^t \]  
(2.9)

\[ \sum_{t=0}^{n} p_{\alpha+n,\beta+r}(x)p_{\gamma+n,\delta-r}(y) \left( \frac{x^2-1}{y^2-1} \right)^t \]  
(2.10)
\[
(\Delta_x + \Delta_y)^n \left[ \frac{a^x}{\Gamma(x-n+1)} \frac{a^y}{\Gamma(y-n+1)} \right]
\]
\[
= (-1)^n a^{x+y} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{1}{\Gamma(x-r+1)} \frac{1}{\Gamma(y+r+1)} \binom{\alpha}{n-r} \binom{\gamma}{r} \frac{c_r^{(\alpha)}}{c_r^{(\gamma)}}
\]
where \( c_r^{(\alpha)} \) Gegenbauer polynomial.

\[
(\Delta_x + \Delta_y)^n \left[ \frac{c^x \Gamma(x+\beta)}{\Gamma(x-n+1)} \frac{d^y \Gamma(y+r)}{\Gamma(y-n+1)} \right]
\]
\[
= c^{x+y} d^y \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{\Gamma(x-r+\beta)}{\Gamma(x-r+1)} \frac{\Gamma(y+r+\gamma)}{\Gamma(y+r+1)} M_{n-r}(x-r;\beta,c) M_{r}(y+r;\gamma,d)
\]
Where \( M_n(x;\beta,c) \) is denoted by Meixner polynomial.

\[
(\Delta_x + \Delta_y)^n \left[ \frac{\Gamma(x+\alpha) \Gamma(x+\beta)}{\Gamma(x-n+1)} \frac{\Gamma(y+\delta) \Gamma(y+\phi)}{\Gamma(y-n+1) \Gamma(y-n+\psi)} \right]
\]
\[
= n! \sum_{r=0}^{n} \frac{\Gamma(x-r+\alpha) \Gamma(x-r+\beta) \Gamma(y+r+\delta) \Gamma(y+r+\phi)}{\Gamma(x-r+1) \Gamma(y-r+1) \Gamma(y-r+\gamma) \Gamma(y+r+1)}
\times P_{n-r}(x-r;\alpha,\beta,\gamma) P_{r}(y+r;\delta,\phi,\psi)
\]
(2.13)
Where \( P_n(x) \) is the Hahn polynomial. Further generalizations was given in [10-13]

Now by taking special values of \( f(x), g(x) \) and \( h(x) \) in (2.2) and \( f(x), g(y) \) and \( h(z) \) in (2.5), we obtain the following partial differential operator representations of trilateral generating formulae for various polynomials:

Operational representations of trilateral double generating functions using (2.2)

\[
D^n \left[ x^{\alpha+n} e^{-x} H_m(x) P_k(\beta,\gamma)(x) \right] = n! e^{-x} x^\alpha \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{m}{s} \frac{\psi + \beta + \gamma + k + \psi}{r!}
\times \left( \binom{\alpha+s}{n-s} \right) H_{m-s}(x) P_{k-r}(x) (2x)^s \left( \frac{x}{2} \right)^r
\]
(3.1)

\[
D^n \left[ x^{\alpha+n} e^{-x} H_m(x) P_k(\beta,\beta)(x) \right] = n! e^{-x} x^\alpha \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{m}{s} \frac{\psi + 2\beta + k + \psi}{r!}
\times \left( \binom{\alpha+s}{n-s} \right) H_{m-s}(x) P_{k-r}(x) (2x)^s \left( \frac{x}{2} \right)^r
\]
(3.2)

\[
D^n \left[ x^{\alpha+n} e^{-x} H_m(x) P_k(x) \right] = n! e^{-x} x^\alpha \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{m}{s} \frac{\psi + k + \psi}{r!}
\times \left( \binom{\alpha+s}{n-s} \right) H_{m-s}(x) P_{k-r}(x) (2x)^s \left( \frac{x}{2} \right)^r
\]
(3.3)

\[
D^n \left[ x^{\alpha+n} e^{-x} P_m^{(\beta,\beta)}(x) P_k^{(\delta,\delta)}(x) \right] = n! e^{-x} x^\alpha \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{m}{s} \frac{\psi + \beta + \gamma + m + \psi}{r!}
\times \left( \binom{\alpha+s}{n-s} \right) H_{m-s}(x) P_{k-r}(x) (2x)^s \left( \frac{x}{2} \right)^r
\]
\[ \times I_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\beta+\gamma)}(x) P_{k-r}^{(\delta+\epsilon)}(x) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.4)

\[ \times H_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\gamma+\delta)}(x) P_{k-r}^{(\epsilon+\zeta)}(x) (2) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.5)

\[ \times H_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\gamma+\delta)}(x) P_{k-r}^{(\epsilon+\zeta)}(x) (2) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.6)

In particular, if \( m = n = k \) in (2.6)- (2.11) then, we get,

\[ \times H_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\gamma+\delta)}(x) P_{k-r}^{(\epsilon+\zeta)}(x) (2) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.7)

\[ \times H_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\gamma+\delta)}(x) P_{k-r}^{(\epsilon+\zeta)}(x) (2) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.8)

\[ \times H_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\gamma+\delta)}(x) P_{k-r}^{(\epsilon+\zeta)}(x) (2) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.9)

\[ \times H_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\gamma+\delta)}(x) P_{k-r}^{(\epsilon+\zeta)}(x) (2) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.10)

\[ \times H_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\gamma+\delta)}(x) P_{k-r}^{(\epsilon+\zeta)}(x) (2) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.11)

\[ \times H_{n-r-s}^{(\alpha+\beta)}(x) P_{m-s}^{(\gamma+\delta)}(x) P_{k-r}^{(\epsilon+\zeta)}(x) (2) \left( \frac{x}{2} \right)^{r} \left( \frac{x}{2} \right)^{s} \left( \frac{x}{2} \right)^{t} \]  

(3.12)

**Proof:** It is obvious that the above results can prove easily with help of (2.2) and the equations given in section 1 (Equations (1.2)-

---


**5/7**
So we omit the details.

Operational representations of trilateral double generating functions using (2.5)

\[
\left( \begin{array}{c}
D_x + D_y + D_z
\end{array} \right)_n^{(1.5)} e^{-x-y^2} (z-1)^{\beta+n} (z+1)^{\gamma+n} = n! e^{-x-y} x^{\alpha} (z-1)^{\beta} (z+1)^{\gamma}
\]

\[
\sum_{r=0}^{n-r} \sum_{s=0}^{n-r} f_{n-r-s}^{(a+r+s)}(x)H_r(y)P_s^{(\beta+s,y+s)}(z) (-x)^r \left[ \frac{z^2 - 1}{2} \right]^s
\]  

(4.1)

\[
(D_x + D_y + D_z)\left( \begin{array}{c}
D_x + D_y + D_z
\end{array} \right)_n^{(1.5)} e^{-x-y^2} (y-1)^{n+\beta} (y+1)^{n+\gamma} (z^2 - 1)^{\delta+n} = n! e^{-x} x^{\alpha} (y-1)^{\beta} (y+1)^{\gamma} (z^2 - 1)^{\delta}
\]

\[
\sum_{r=0}^{n-r} \sum_{s=0}^{n-r} L_{n-r-s}^{(a+r+s)}(x)H_r(y)P_s^{(\beta+r,y+r)}(y)P_{\delta+r+s}^{(\gamma+s,\gamma+s)}(z) \left[ \frac{y^2 - 1}{2} \right]^r \left[ \frac{z^2 - 1}{2} \right]^s
\]

(4.3)

\[
(D_x + D_y + D_z)_n^{(1.5)} e^{-x^2} (y-1)^{n+\alpha} (y+1)^{n+\beta} (z^2 - 1)^{\gamma+n} = (-1)^n e^{-x^2} (y-1)^{n+\alpha} (y+1)^{n+\beta} (z^2 - 1)^{\gamma}
\]

\[
\sum_{r=0}^{n-r} \sum_{s=0}^{n-r} (-n)_{r-s} H_{n-r-s}(x)P_{r+s}^{(\alpha+r+s,y+r+s)}(z) \left[ \frac{y^2 - 1}{2} \right]^r \left[ \frac{z^2 - 1}{2} \right]^s
\]

(4.4)

\[
(D_x + D_y + D_z)_n^{(1.5)} e^{-x} (y-1)^{\beta} (y+1)^{\gamma} (z^2 - 1)^{\delta+n} = \sum_{r=0}^{n-r} \sum_{s=0}^{n-r} \left( \begin{array}{c}
m_r
\end{array} \right) \frac{(1+\beta+\gamma+k)_s}{s!}
\]

(4.5)

\[
(D_x + D_y + D_z)_n^{(1.5)} e^{-x} (y-1)^{\beta} (y+1)^{\gamma} (z^2 - 1)^{\delta+n} = \sum_{r=0}^{n-r} \sum_{s=0}^{n-r} \left( \begin{array}{c}
m_r
\end{array} \right) \frac{(1+2\beta+k)_s}{s!}
\]

(4.6)
\[
\times L^{(\alpha+\tau)}_{n-r-s}(x)P_{m-r-s}(y)P_{k-s}(z)\left(\frac{x}{2}\right)^{\tau}s \tag{4.7}
\]

\[
(D_x + D_y + D_z)^n \frac{x^\alpha e^{-x} P_{m-r-s}^{(\beta,\gamma)}(y)P_{k-s}^{(\delta,\delta)}(z)}{r!s!} = n!e^{-x}x^\alpha \tag{4.8}
\]

\[
(D_x + D_y + D_z)^n \frac{x^{\alpha+n} e^{-x} P_{m-r-s}^{(\beta,\gamma)}(y)P_{k-s}^{(\delta,\delta)}(z)}{r!s!} = n!e^{-x}x^\alpha \tag{4.9}
\]

\[
(D_x + D_y + D_z)^n \frac{(-1)^n e^{-x^2}}{r!s!} \times H_{n-r-s}(x)P_{m-r-s}^{(\alpha+\tau,\beta+\tau)}(y)P_{k-s}^{(\gamma+\tau,\gamma+\tau)}(z)\left(\frac{1}{2}\right)^{r+s} \tag{4.10}
\]

**Proof:** It is noticeable that the above results can prove easily with help of (2.2), (2.3), (2.4) and the equations (1.2)-(1.5). So we omit the details.

**CONCLUDING REMARKS**

In sections 3 and 4, we have obtained a new class of trilateral generating function in the form of equation (2.2) and (2.5) respectively. By selecting suitable values of different parameters in equations (3.1) to (4.10), one can easily derive a new class of bilateral generating functions that involves various polynomials that will be the particular case of present study.

**REFERENCES**


**Cite this article**