INTRODUCTION

Consider the data in (Table 1), taken directly from Stuart [1], which is the data on unaided distance vision of 7477 women aged 30 to 39 employed in Royal Ordnance factories in Britain from 1943 to 1946.

For the vision data, since many observations concentrate on (or near) the main diagonal cells in the table, a woman’s right eye grade is strongly associated with her left eye grade.

So, we are interested in whether a woman’s right eye grade is symmetric to her left eye grade or not. Many statisticians have proposed many statistical models of symmetry and asymmetry.

For example, the symmetry (S) model, which represents the symmetry on cell probabilities, and the quasi-symmetry (QS) model, which indicates the symmetry on odds ratios, were proposed by Bowker [2] and Caussinus [3], respectively. Also, as asymmetric models, the linear diagonals-parameter symmetry (LDP) model, which is a special case of the QS model, the extended QS (EQS) model, which is an extension of the QS model, and the generalized LDPS model, which is an extension of the LDP model and is also a special case of the EQS model, were proposed by Agresti [4], Tomizawa [5] and Yamamoto and Tomizawa [6], respectively.

In addition, Tomizawa [7] gave a decomposition of the LDPS model by introducing the diagonal weighted marginal homogeneity (DWMH) I (and the DWMH-II) model as follows: the LDPS model holds if and only if both the QS and the DWMH-I (or the DWMH-II) models hold.

(Table 2) shows the results of goodness-of-fit test of models (including some new models being proposed in this paper). We see from (Table 2) that all the existing LDPS, QS, DWMH-I and DWMH-II models fit these data well. In addition, the DWMH-I model is the best fitting model among these existing models because it has a minimum AIC+ value (for the details of goodness-of-fit test and model selection, see Section Goodness-of-fit test and model selection). Therefore, the DWMH-I model is preferable to the LDPS model, which is the simplest among them. However, we cannot know why this fact occurs even if we use the theorem given by Tomizawa [7].

We can also see that the generalized LDPS (i.e., LDPS (K) (K = 1…, 5)) model fits these data moderately, but the EQS model, which is an extension of the generalized LDPS model, gives poor fit to them. So, we want to investigate these vision data in more detail including the fact why the above occurs by considering a decomposition of the generalized LDPS model using the EQS model. Thus we have to introduce some new models.

Section Models of symmetry and asymmetry reviews some existing statistical symmetry and asymmetry models, Section Decompositions of generalized LDPS model proposes some new models and gives a decomposition of the generalized LDPS model, Section Goodness-of-fit test and model selection describes goodness-of-fit test and model selection, and Section Analysis of unaided vision data shows results of the analyses of two kinds of vision data.
Table 1: Unaided distance vision of 7477 women aged 30-39 employed in Royal Ordnance factories in Britain from 1943 to 1946; from Stuart [1]. (The parenthesized values are maximum likelihood estimates of expected frequencies under the DWMH-II (4) model).

<table>
<thead>
<tr>
<th>Left eye grade</th>
<th>Right eye grade</th>
<th>Best (1)</th>
<th>Second (2)</th>
<th>Third (3)</th>
<th>Worst (4)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best (1)</td>
<td></td>
<td>1520</td>
<td>266</td>
<td>124</td>
<td>66</td>
<td>1976</td>
</tr>
<tr>
<td>(1520.00)</td>
<td>(265.68)</td>
<td>(123.90)</td>
<td>(66.00)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Second (2)</td>
<td></td>
<td>234</td>
<td>1512</td>
<td>432</td>
<td>78</td>
<td>2256</td>
</tr>
<tr>
<td>(234.32)</td>
<td>(1512.00)</td>
<td>(432.14)</td>
<td>(78.09)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Third (3)</td>
<td></td>
<td>117</td>
<td>362</td>
<td>1772</td>
<td>205</td>
<td>2456</td>
</tr>
<tr>
<td>(117.12)</td>
<td>(361.86)</td>
<td>(1772.00)</td>
<td>(205.18)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Worst (4)</td>
<td></td>
<td>36</td>
<td>82</td>
<td>179</td>
<td>492</td>
<td>789</td>
</tr>
<tr>
<td>(36.00)</td>
<td>(81.88)</td>
<td>(178.82)</td>
<td>(492.00)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>1907</td>
<td>2222</td>
<td>2507</td>
<td>841</td>
<td>7477</td>
</tr>
</tbody>
</table>

Table 2: Values of likelihood ratio chi-squared statistic $G^2$ and AIC* for models applied to the data in Table 1.

<table>
<thead>
<tr>
<th>Models</th>
<th>Degrees of freedom</th>
<th>$G^2$</th>
<th>$p$-value</th>
<th>AIC*</th>
</tr>
</thead>
<tbody>
<tr>
<td>LDPS (i.e., LDPS(0))</td>
<td>5</td>
<td>7.28</td>
<td>0.201</td>
<td>-2.72</td>
</tr>
<tr>
<td>LDPS(1)</td>
<td>5</td>
<td>6.83</td>
<td>0.233</td>
<td>-3.17</td>
</tr>
<tr>
<td>LDPS(2)</td>
<td>5</td>
<td>6.85</td>
<td>0.232</td>
<td>-3.15</td>
</tr>
<tr>
<td>LDPS(3)</td>
<td>5</td>
<td>6.91</td>
<td>0.228</td>
<td>-3.09</td>
</tr>
<tr>
<td>LDPS(4)</td>
<td>5</td>
<td>6.96</td>
<td>0.223</td>
<td>-3.04</td>
</tr>
<tr>
<td>LDPS(5)</td>
<td>5</td>
<td>7.01</td>
<td>0.220</td>
<td>-2.99</td>
</tr>
<tr>
<td>QS</td>
<td>3</td>
<td>7.27</td>
<td>0.064</td>
<td>1.27</td>
</tr>
<tr>
<td>EQS</td>
<td>2</td>
<td>6.82</td>
<td>0.033</td>
<td>2.82</td>
</tr>
<tr>
<td>DWMH-I (i.e., DWMH-I(0))</td>
<td>2</td>
<td>5.23×10^{-3}</td>
<td>0.997</td>
<td>-3.9948</td>
</tr>
<tr>
<td>DWMH-I(1)</td>
<td>2</td>
<td>2.51×10^{-3}</td>
<td>0.999</td>
<td>-3.9975</td>
</tr>
<tr>
<td>DWMH-I(2)</td>
<td>2</td>
<td>2.45×10^{-3}</td>
<td>0.999</td>
<td>-3.9976</td>
</tr>
<tr>
<td>DWMH-I(3)</td>
<td>2</td>
<td>2.67×10^{-3}</td>
<td>0.999</td>
<td>-3.9973</td>
</tr>
<tr>
<td>DWMH-I(4)</td>
<td>2</td>
<td>2.89×10^{-3}</td>
<td>0.999</td>
<td>-3.9971</td>
</tr>
<tr>
<td>DWMH-I(5)</td>
<td>2</td>
<td>3.08×10^{-3}</td>
<td>0.998</td>
<td>-3.9969</td>
</tr>
<tr>
<td>DWMH-II (i.e., DWMH-II(0))</td>
<td>2</td>
<td>0.02</td>
<td>0.993</td>
<td>-3.98</td>
</tr>
<tr>
<td>DWMH-II(1)</td>
<td>2</td>
<td>4.37×10^{-3}</td>
<td>0.998</td>
<td>-3.9956</td>
</tr>
<tr>
<td>DWMH-II(2)</td>
<td>2</td>
<td>2.24×10^{-3}</td>
<td>0.999</td>
<td>-3.9978</td>
</tr>
<tr>
<td>DWMH-II(3)</td>
<td>2</td>
<td>1.77×10^{-3}</td>
<td>0.999</td>
<td>-3.9982</td>
</tr>
<tr>
<td>DWMH-II(4)</td>
<td>2</td>
<td>1.75×10^{-3}</td>
<td>0.999</td>
<td>-3.9983</td>
</tr>
<tr>
<td>DWMH-II(5)</td>
<td>2</td>
<td>1.87×10^{-3}</td>
<td>0.999</td>
<td>-3.9981</td>
</tr>
<tr>
<td>2PA-I(1)</td>
<td>1</td>
<td>1.21</td>
<td>0.271</td>
<td>-0.79</td>
</tr>
<tr>
<td>2PA-I(2)</td>
<td>1</td>
<td>0.74</td>
<td>0.390</td>
<td>-1.26</td>
</tr>
<tr>
<td>2PA-I(3)</td>
<td>1</td>
<td>0.53</td>
<td>0.467</td>
<td>-1.47</td>
</tr>
<tr>
<td>2PA-I(4)</td>
<td>1</td>
<td>0.42</td>
<td>0.519</td>
<td>-1.58</td>
</tr>
<tr>
<td>2PA-I(5)</td>
<td>1</td>
<td>0.35</td>
<td>0.556</td>
<td>-1.65</td>
</tr>
<tr>
<td>2PA-II(1)</td>
<td>1</td>
<td>1.25</td>
<td>0.264</td>
<td>-0.75</td>
</tr>
<tr>
<td>2PA-II(2)</td>
<td>1</td>
<td>0.77</td>
<td>0.381</td>
<td>-1.23</td>
</tr>
<tr>
<td>2PA-II(3)</td>
<td>1</td>
<td>0.55</td>
<td>0.458</td>
<td>-1.45</td>
</tr>
<tr>
<td>2PA-II(4)</td>
<td>1</td>
<td>0.43</td>
<td>0.510</td>
<td>-1.57</td>
</tr>
<tr>
<td>2PA-II(5)</td>
<td>1</td>
<td>0.36</td>
<td>0.548</td>
<td>-1.64</td>
</tr>
</tbody>
</table>
282] and Agresti [9, p. 426]. This indicates that the probability that an observation will fall in the $(i, j)$ cell, $i \neq j$ is equal to the probability that the observation falls in the symmetric $(j, i)$ cell.

Caussinus [3] considered the QS model defined by

$$p_{ij} = \mu_{i\alpha} \beta_{j\psi}, (i = 1, \ldots, R; j = 1, \ldots, R).$$

Where $\psi_1 = \psi_p$. (Also see Bishop et al. [8, p. 286]) Tomizawa [5] considered the EQS model defined by

$$p_{ij} = \mu_{i\alpha} \beta_{j\psi}, (i = 1, \ldots, R; j = 1, \ldots, R).$$

Where $\psi_1 = \psi_p (i < j)$. Note that this model may be expressed as

$$p_{ij} = \gamma_{i\alpha} p_{ip} (i < j).$$

A special case of this model obtained by putting $\gamma = 1$ is the QS model.

Agresti [4] considered the LDPS model defined by

$$p_{ij} = \psi_{i\alpha}(v_{ij}).$$

Where $\psi_1 = \psi_p$. A special case of this model obtained by putting $\theta = 1$ is the S model. The LDPS model is a special case of the QS model. Yamamoto and Tomizawa [6] considered the generalized LDPS (LDPS $(K)$) model as follows: for a fixed $K = 0,1,2,\ldots$,

$$p_{ij} = \psi_{i\alpha}(v_{ij}).$$

Where $\psi_1 = \psi_p$. Especially the LDPS $(0)$ model is equivalent to the LDPS model. For the LDPS $(K)$ model, see also Yamamoto, Ohama and Tomizawa [10].

Tomizawa [7] considered the DWMH-I model defined by

$$p_{ij}^\theta (\theta) + p_{ij} = p_{ij} + p_{ij} (i = 1, \ldots, R),$$

Where $\theta$ is unspecified and

$$p_{ij}^\theta (\theta) = \sum_{k=1}^{\theta} \theta^k p_{ij}, p_{ij} = \sum_{k=1}^{\theta} p_{ij}.$$

This model indicates that the row marginal totals summed by multiplying the probabilities $p_{ij}$ for the cells with a distance $i-j (>0)$ below main diagonal in the table by a common weight $\theta^k$ are equal to the column marginal totals summed by the same way. Tomizawa [7] also considered the DWMH-II model defined by

$$p_{ij}^\theta (\theta) + p_{ij} = p_{ij} (i = 1, \ldots, R),$$

Where $\theta$ is unspecified and

$$p_{ij}^\theta (\theta) = \sum_{k=1}^{\theta} \theta^k p_{ij}, p_{ij} = \sum_{k=1}^{\theta} p_{ij}.$$

This model states, in contrast with the case of DWMH-I model, that the row marginal totals summed by multiplying the probabilities $p_{ij}$ for the cells with a distance $j-i (>0)$ above main diagonal in the table by a common weight $\theta^{i-j}$ are equal to the column marginal totals summed by the same way. Note that the DWMH-I (DWMH-II) model with $\theta = 1$ is the marginal homogeneity model (Bishop et al. [8, p. 282]).

Tomizawa [7] gave decompositions of the LDPS model as follows (see also Yamamoto, Shinoda and Tomizawa [11]):

**Theorem 1:** For $t = 1$ and II, the LDPS model holds if and only if both the QS and the DWMH-$t$ models hold.

**Decompositions of generalized LDPS model**

To consider decompositions of the LDPS $(K)$ model, we shall introduce four kinds of new models. First, consider a model as follows: for a fixed $K = 0,1,2,\ldots$,

$$\theta^k p_{ij}^\theta (\theta) + p_{ij} + p_{ij} = p_{ij}^\theta (\theta) + p_{ij} + \theta^k p_{ij} (i = 1, \ldots, R),$$

Where $\theta$ is unspecified. This model indicates that the row marginal totals summed by multiplying the probabilities $p_{ij}$ for the cells with a distance $i-j (>0)$ below main diagonal in the table by a common weight $\theta^k$ are equal to the column marginal totals summed by the same way. When $K = 0$ this model is equivalent to the DWMH-I model. Thus, we will refer to this model as the generalized diagonal weighted marginal homogeneity I (DWMH-I $(K)$) model.

Let $X$ and $Y$ denote the row and column variables, respectively and let $F^X_k = \Pr(X \leq k)$ and $F^Y_k = \Pr(Y \leq k)$ for $k = 1, \ldots, R - 1$. Then we see that

$$F^X_k - F^Y_k = \sum_{r=1}^{k} \sum_{r=1}^{k} p_{i}\sum_{r=1}^{k} p_{i} (k = 1, \ldots, R - 1).$$

Also, under the DWMH-I $(K)$ model, the following equation is satisfied: for $k = 1, \ldots, R - 1$,

$$\sum_{r=1}^{k} \theta^k p_{ij}^\theta (\theta) + p_{ij} + p_{ij} = \sum_{r=1}^{k} \theta^k p_{ij}^\theta (\theta).$$

So, from the above equation, we see

$$\sum_{r=1}^{k} \sum_{r=1}^{k} p_{i}\sum_{r=1}^{k} p_{i} (k = 1, \ldots, R - 1).$$

Therefore, we can see for $k = 1, \ldots, R - 1$,

$$F^X_k - F^Y_k = \sum_{r=1}^{k} \theta^k \sum_{r=1}^{k} \theta^k p_{ij} - \sum_{r=1}^{k} \sum_{r=1}^{k} p_{i}.$$

Thus, under the DWMH-I $(K)$ $(K = 0,1,2,\ldots)$, model, we see that $\theta \geq 1$ is equivalent to $F^X_k \geq F^Y_k$ for $k = 1, \ldots, R - 1$. So, the parameter $\theta$ in the DWMH-I $(K)$ model would be useful for making inferences such as that $X$ is stochastically less than $Y$ or vice versa.

Secondly, consider a model as follows: for a fixed $K$
Central this. are equal to the column denotes the transpose and the ( ) the LDPS ( ) model. Therefore, noting that 

consider a model defined by satisfies ( ) and using, ( ) as follows:

where matrix in the EQS model is equal to is unspecified. This model states that the row model, we can obtain the we see for we see consider a model defined by where

would be useful to consider the decomposition for the LDPS ( ) model for the data, but the 2PA-I ( ) model. It may be not meaningful to apply K agement I (2PA-I ( )) model. We shall refer to this model as the two-parameter K model. Note K DWMH-II ( ) model when both the EQS and DWMH-II ( ) models hold. We shall refer to this model as the two-parameter K model.

K model is equivalent to the DWMH-II model. Thus, we will refer to this model as the two-parameter K model. Note that the 2PA- K model holds if and only if all the EQS, DWMH- K model holds and then we shall show that the LDPS ( ) model holds.

First, consider the case of . From the assumption that the EQS model holds, we obtain

\[ \gamma \rho_i = \mu \theta \rho_i, \quad (i = 1, \ldots, R). \]

Where ( ) is unspecified. This model states that the row marginal totals summed by multiplying the probabilities \( p_{ij} \) for the cells with a distance \( j-i > 0 \) above main diagonal in the table by a common weight \( \theta \) are equal to the column marginal totals summed by the same way. When \( K = 0 \) this model is equivalent to the DWMH-II model. Thus, we will refer to this model as the generalized diagonal weighted marginal homogeneity II (DWMH-II (K)) model.

For the DWMH-II (K) ( \( K = 0,1,2, \ldots \)), model, we can obtain the similar property to the DWMH-I (K) model.

Let \( \gamma = \sum_{i=1}^{K} p_i p_i p_i \).

Thirdly, for a fixed \( K = 1,2, \ldots \), consider a model defined by \( \gamma = \theta^k \), where ( ) satisfies

\[ p_i = \theta^k p_i, \quad (\theta). \]

This indicates that \( \gamma \) in the EQS model is equal to \( \theta^k \) in the DWMH-I (K) model when both the EQS and DWMH-I (K) models hold. We shall refer to this model as the two-parameter agreement I (2PA-I (K)) model. It may be not meaningful to apply only the 2PA-I (K) model for the data, but the 2PA-I (K) model would be useful to consider the decomposition for the LDPS (K) model (see Section Analysis of unaided vision data).

Finally, for a fixed \( K = 1,2, \ldots \), consider a model defined by

\[ \frac{1}{\gamma} - \theta^k \]

where ( ) satisfies

\[ \theta^k p_{ij} (\theta) = p_{ij}. \]

This indicates that \( \gamma \) in the EQS model is equal to \( \theta^k \) in the DWMH-II (K) model when both the EQS and DWMH-II (K) models hold. We shall refer to this model as the 2PA-II (K) model. Note that the 2PA-II (K) model ( \( t = 1,2, \ldots \)) does not include the case of \( K = 0 \).

Then we obtain decompositions of the LDPS (K) model ( \( K = 1,2, \ldots \)) as follows:

**Theorem 2:** For \( t = 1,2, \ldots \) and fixed \( K = 1,2, \ldots \) the LDPS (K) model holds if and only if all the EQS, DWMH-t (K) and 2PA-t (K) models hold.

***Proof:*** For \( t = 1,2, \ldots \), if the LDPS (K) model holds, then all the EQS, DWMH-t (K) and 2PA-t (K) models hold. Conversely, assuming that all the EQS, DWMH-t (K) and 2PA-t (K) models hold and then we shall show that the LDPS (K) model holds.

First, consider the case of \( t = 1 \). From the assumption that the EQS model holds, we obtain

\[ \gamma \rho_i = \mu \theta \rho_i, \quad (i = 1, \ldots, R). \]

Where

\[ \mu = \sum_{i=1}^{K} \theta^k \beta \rho_{ij} + \sum_{i=1}^{K} \beta \rho_{ij}. \]

Also we obtain

\[ \rho_i = \gamma \theta \rho_i, \quad (\theta) \]

where

\[ \gamma = \prod_{i=1}^{K} \rho_i (\theta). \]

Since (1) is equal to (2) in terms of the assumption that the DWMH-I (K) model holds, we obtain

\[ \alpha_i = \tau_i \beta, \quad (i = 1, \ldots, R), \]

where

\[ h_i = \frac{\gamma}{\mu}. \]

By substituting (3) in \( \gamma \) and using, \( \gamma = \tau \theta^k \), we see

\[ \nu_i = \sum_{i=1}^{K} \theta^k h_i \beta \rho_{ij} + h_i \beta \rho_{ij} + \sum_{i=1}^{K} \theta^k h_i \beta \rho_{ij}. \]

Since the 2PA-I (K) model holds, i.e., \( \gamma = \theta^k \) we see

\[ \nu_i = \sum_{i=1}^{K} \theta^k h_i \beta \rho_{ij} + h_i \beta \rho_{ij} + \sum_{i=1}^{K} \theta^k h_i \beta \rho_{ij}. \]

Therefore, since (4), we see

\[ f = Wf, \]

where

\[ f = (h_1, \ldots, h_N)^T \]

And “r” denotes the transpose and the \( (i, s)^{th} \) element of the \( R \times R \) matrix \( W \) is given by

\[ (W)_{ij} = \begin{cases} \frac{1}{\mu} \theta^k \beta \rho_{is} (i > s), \\ \frac{1}{\mu} \beta \rho_{is} (i = s), \\ \frac{1}{\mu} \theta^k \beta \rho_{is} (i < s). \end{cases} \]

All elements of \( W \) are positive and satisfy \( WJ = Jf \) where \( J = (1,1, \ldots, 1)^T \). Therefore, noting that \( \{ h_i > 0 \} \), we obtain \( f = CJ \), where \( C \) is a constant. Thus, from (3) we obtain

\[ \frac{\alpha_i}{\beta_i} = \alpha \theta^{-i} (i = 1, \ldots, R). \]

Noting that \( \psi_j = \gamma \rho_{ij} = \theta^k \rho_{ij} \), we see for \( i < j \)

\[ \frac{\alpha_i}{\beta_i} \theta^k \rho_{ij} = \theta^k (\theta^{-i-j}). \]

Thus the LDPS (K) model holds. The case of \( t = 1 \) can also be
proved in the same way. The proof is completed.

Note that Theorem 2 does not include the case of $K = 0$.

**Goodness-of-fit test and model selection**

Let $n_{ij}$ denote the observed frequency in the $i$th row and $j$th column of the $R \times R$ table with $n = \sum n_{ij}$, and let $\hat{m}_{ij}$ denote the corresponding expected frequency. Assume that $\{n_{ij}\}$ have a multinomial distribution. The maximum likelihood estimates (MLEs) of expected frequencies $\{\hat{m}_{ij}\}$ under each model could be obtained, for example, using the Newton-Raphson method to the log-likelihood equations. Each model can be tested for goodness-of-fit by the likelihood ratio chi-squared statistic $G^2$ with the corresponding degrees of freedom, defined by

$$G^2 = 2 \sum_{i=1}^{R} \sum_{j=1}^{R} n_{ij} \log \left( \frac{n_{ij}}{\hat{m}_{ij}} \right)$$

Where $\hat{m}_{ij}$ is the MLE of $m_{ij}$ under the model. The numbers of degrees of freedom for the LDPS ($K$), EQS, DWMH-$t$ ($K$), and $2PA\,-\,t$ ($K$) ($t = I, II$) models are $(R - 2)(R + 1)/2$, $R(R - 3)/2$, $R - 2$, and 1, respectively. Note that the number of degrees of freedom for the LDPS ($K$) model is equal to the sum of those for the decomposed models.

A quick method for choosing the best-fitting model among different models which include non-nested models is to use Akaike's [12] information criterion (AIC), which is defined as

$$AIC = -2(\text{maximum log-likelihood}) + 2(\text{number of free parameters})$$

for each model (Konishi and Kitagawa [13, p. 61]). This criterion gives the best-fitting model as the one with minimum AIC. Since only the difference between AIC's is required when two models are compared, it is possible to ignore a common constant of AIC and we may use a modified AIC defined as

$$AIC^* = G^2 - 2(\text{number of degrees of freedom}).$$

Thus, for the data, the model with the minimum $AIC^*$ (i.e., the minimum AIC) is the best-fitting model.

**Analysis of unaided vision data**

Consider the vision data in (Table 1) again. We see from (Table 2) that the new models DWMH-I ($K$), DWMH-II ($K$), $2PA\,-\,I$ ($K$) and $2PA\,-\,II$ ($K$) ($K = 1, \ldots, 5$) give good fits to these data. In addition, we see that the DWMH-II ($4$) model has a minimum $AIC^*$ value (with the $p$-value 0.999). Thus, the DWMH-II ($4$) model is the best fitting model among models considered here. Therefore the DWMH-II ($4$) model, which is one of decomposed models of the LDPS ($4$) model (Theorem 2), is preferable to the LDPS ($4$) model.

Under the DWMH-II ($4$) model, the MLE of $\boldsymbol{\theta}$, i.e., $\hat{\theta}$ is 0.97. Since $\hat{\theta} < 1$, under this model, the marginal probability that the right eye grade is $i$ or below ($i = 1, 2, 3$) is estimated to be greater than the marginal probability that the left eye grade is $i$ or below, namely a woman's right eye is estimated to be better than her left eye.

We shall show another example of vision data. Consider the data in (Table 3), taken directly from Tomizawa [5], are constructed from unaided distance vision of 4746 students aged 18 to about 25 including about 10% women in Faculty of Science and Technology, Science University of Tokyo, Japan, examined in April 1982.

We see from (Table 4) that all the models fit these data well and the LDPS ($5$) model is the best fitting model among all the models. However, since the new models DWMH-I ($K$) and DWMH-II ($K$) ($K = 1, \ldots, 5$) give an easy and simple interpretation for the data, we will consider especially these models here. Among the DWMH-I ($K$) and DWMH-II ($K$) ($K = 1, \ldots, 5$) models, we see from the $AIC^*$ value in (Table 4) that the DWMH-II ($5$) model is the most preferable model (with the $p$-value 0.714). Under the DWMH-II ($5$) model, $\hat{\theta}$ is 1.03. Thus, under this model, the marginal probability that the left eye grade is $i$ or below ($i = 1, 2, 3$) is estimated to be greater than the marginal probability

<table>
<thead>
<tr>
<th>Right eye grade</th>
<th>Best (1)</th>
<th>Second (2)</th>
<th>Third (3)</th>
<th>Worst (4)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best (1)</td>
<td>1291</td>
<td>130</td>
<td>40</td>
<td>22</td>
<td>1483</td>
</tr>
<tr>
<td>(1291.00)</td>
<td>(130.23)</td>
<td>(38.63)</td>
<td>(21.89)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Second (2)</td>
<td>149</td>
<td>221</td>
<td>114</td>
<td>23</td>
<td>507</td>
</tr>
<tr>
<td>(148.79)</td>
<td>(221.00)</td>
<td>(110.04)</td>
<td>(22.85)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Third (3)</td>
<td>64</td>
<td>124</td>
<td>660</td>
<td>185</td>
<td>1033</td>
</tr>
<tr>
<td>(65.86)</td>
<td>(127.79)</td>
<td>(660.00)</td>
<td>(190.65)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Worst (4)</td>
<td>20</td>
<td>25</td>
<td>249</td>
<td>1429</td>
<td>1723</td>
</tr>
<tr>
<td>(20.08)</td>
<td>(25.13)</td>
<td>(243.06)</td>
<td>(1429.00)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1524</td>
<td>500</td>
<td>1063</td>
<td>1659</td>
<td>4746</td>
</tr>
</tbody>
</table>
that the right eye grade is \( i \) or below, namely a student’s left eye is estimated to be better than his/her right eye. This is contrast to the result of women’s vision data analysis.

**CONCLUSION**

We have proposed the DWMH-\( t \) (\( t = I, II \)) models and have given Theorem 2, which is the decompositions of the LDPS (\( K \)) model. Theorem 2 would be useful for exploring the reason for the poor fit not only when the LDPS (\( K \)) model fits the data poorly, but even if there is a preferable model to the LDPS (\( K \)) model such as the case of the analysis for (Table 1). In addition, as described in Section Decompositions of generalized LDPS model, the parameter \( \theta \) in the DWMH-\( t(K) \) (\( t = I, II \)) model would be useful for making inferences such as that \( X \) is stochastically less than \( Y \) or vice versa and this has been shown in Section Analysis of unaided vision data via vision data analyses.

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