The Arithmetic-Geometric-Harmonic Mean

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Abstract
The arithmetic-geometric mean of two positive numbers has been defined to be the common limit of the sequences \( \{a_n\} \) and \( \{b_n\} \) defined recursively by
\[
a_{n+1} = \left( \frac{a_n + b_n}{2} \right)^2, \quad n = 0, 1, 2, \ldots; \quad a_0 = a, b_0 = b.
\]
This mean has applications to elliptic integrals. The basic idea of this paper is to generalize the above mean to one that uses three numbers by using the harmonic mean to define a third sequence as proposed by Dalpatadu and use elementary techniques to prove the existence of such a mean and obtain results on rates of convergence.

INTRODUCTION
The arithmetic-geometric mean of two positive numbers \( a \leq b \), denoted by \( \text{AGM}(a, b) \), is defined to be the common limit of the sequences \( \{a_n\} \) and \( \{b_n\} \) defined recursively by [1]
\[
a_{n+1} = \left( \frac{a_n + b_n}{2} \right)^2, \quad n = 0, 1, 2, \ldots; \quad a_0 = a, b_0 = b.
\]
It has also been shown that [2] the reciprocal of this mean equals
\[
\frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}.
\]
We wish to generalize the above mean and explore the possibility of evaluating integrals such as the ellipsoidal surface integral
\[
\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \frac{d\phi d\theta}{\left[ a^2 \cos^2 \theta \sin^2 \phi + b^2 \sin^2 \theta \cos^2 \phi + c^2 \cos^2 \theta \sin^2 \phi \right]^{3/2}}.
\]
In this paper, we will define the arithmetic-geometric-harmonic mean of three positive numbers \( a \leq b \leq c \) via three sequences \( \{a_n\}, \{b_n\}, \{c_n\} \). We shall establish the existence of such a mean and obtain a couple of results on the convergence of these sequences. A relationship between this mean and the above surface integral has not yet been established. We will show that a natural generalization such as the one suggested above is not the proper application of the arithmetic-geometric-harmonic mean.

THE ARITHMETIC-GEOMETRIC-HARMONIC MEAN
In this section, the arithmetic-geometric-harmonic mean is defined and its existence is proved. Furthermore, two important results on rates of convergence are also established.

Definition 1
We will first define the three sequences using the arithmetic, geometric, and harmonic means.

Let \( 0 < a \leq b \leq c \). Define \( \{a_n\}, \{b_n\}, \{c_n\} \) recursively by
\[
a_{n+1} = \left( \frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n} \right)^{-1}, \quad n = 0, 1, 2, \ldots; \quad a_0 = a, b_0 = b.
\]
The first of the propositions deals with the ordering of the three sequences.

**Proposition 1:** \(0 < a_n \leq b_n \leq c_n, n = 0,1,2,...\)

**Proof:** The result holds if \(n = 0\). Suppose \(0 < a_n \leq b_n \leq c_n\) for some non negative integer \(n\). Then \(a_{n+1} \leq b_{n+1} \leq c_{n+1}\), because harmonic mean \(\leq\) geometric mean \(\leq\) arithmetic mean of three positive number. Furthermore, \(a_{n+1} > 0\). Hence result.

The next proposition deals with the monotonicity of the sequence obtained via the harmonic mean.

**Proposition 2:** \(a_{n+1} \geq a_n, n = 0,1,2,...\)

**Proof:**

\[
\begin{align*}
\frac{a_{n+1}}{a_n} &= \frac{\frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n}}{\frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n}} \\
&= \frac{1}{b_n + c_n} \cdot \left(\frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n}\right) \\
&\geq 1 \quad \text{for } n = 0,1,2,...
\end{align*}
\]

The result follows because \(0 < a \leq b \leq c\), implies that

\[
\begin{align*}
(b \cdot \frac{1}{a}) - a^2 &\geq 0 \\
\text{with equality if and only if } a = b = c
\end{align*}
\]

We need the next proposition in order to establish the monotonicity of the second sequence.

**Proposition 3:** \(0 < a \leq b \leq c\). Then \(ac \leq b^2\) if and only if \(ac \leq b^2\).

**Proof:** Let \(0 < a \leq b \leq c\). Then \(ac \leq b^2\) if and only if \(ac \leq b^2\).

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\[
\begin{align*}
(b \cdot \frac{1}{a}) - a^2 &\geq 0 \\
\text{with equality if and only if } a = b = c
\end{align*}
\]

We need the next proposition in order to establish the monotonicity of the second sequence.

**Proposition 4:** \(0 < a \leq b \leq c\). Then \(ac \leq b^2\) if and only if \(ac \leq b^2\).

**Proof:**

\[
\begin{align*}
\frac{a_{n+1} - b_{n+1}}{a_n - b_n} &= \frac{a_{n+1} + b_{n+1} + c_{n+1}}{a_n + b_n + c_n} \cdot \left(\frac{1}{a_{n+1} + b_{n+1} + c_{n+1}}\right)^{\frac{1}{n}} \\
&= \frac{1}{a_{n+1} + b_{n+1} + c_{n+1}} \cdot \left(\frac{1}{a_n + b_n + c_n}\right)^{\frac{1}{n}} \\
&\geq 1 \quad \text{for } n = 0,1,2,...
\end{align*}
\]

The result follows easily when we use mathematical induction along with the above lemma.

The following proposition proves the monotonicity of the sequence defined as the arithmetic mean.
Proposition 5: $c_{n+1} \leq c_n$, $n = 0, 1, 2, \ldots$

Proof: $c_{n+1} = \frac{a_n + b_n + c_n}{3} \leq \frac{c_n + c_n + c_n}{3} = c_n.$

Remark 1: \{a_n\} is an increasing bounded sequence, \{b_n\} is a monotonic bounded sequence, and \{c_n\} is a decreasing bounded sequence. Thus $\lim_{n \to \infty} a_n, \lim_{n \to \infty} b_n$, and $\lim_{n \to \infty} c_n$ exist. Let these limits be $A$, $B$, and $C$, respectively.

Proposition 6: $A = B = C$.

Proof: By Proposition 1, $A \leq B \leq C$.

$$c_{n+1} = \frac{a_n + b_n + c_n}{3} \implies C = \frac{A + B + C}{3}.$$ Therefore, $A + B \geq 2C$ and $A \geq B$. Since $A \leq B$, $A = B$ and it follows that $A = C$.

Definition 2

Let $0 < a \leq b \leq c$. The Arithmetic-Geometric-Harmonic Mean of the numbers $a$, $b$, and $c$, denoted by AGHM $(a, b, c)$, is the common limit of the sequences $\{a_n\}, \{b_n\},$ and $\{c_n\}$.

We finally end this section by establishing two results on rates of convergence.

Proposition 7: $c_n - a_n \leq \left(\frac{2}{3}\right)^n (c - a)$, $n = 1, 2, \ldots$

Proof: $c_n - a_n \leq c_n - a_{n-1}$ (Proposition 2)

$$= \frac{a_{n-1} + b_{n-1} + c_{n-1}}{3} - a_{n-1} = \frac{b_{n-1} + c_{n-1} - 2a_{n-1}}{3} \leq \frac{2c_{n-1} - 2a_{n-1}}{3} \quad \text{(Proposition 1)}$$

$$= \frac{2}{3}(c_{n-1} - a_{n-1}).$$

Mathematical induction with the above inequality will give us the desired result.

Remark 2: Proposition 7 indicates that we have fairly rapid convergence.

Proposition 8: $c_n - a_n \leq \frac{(c_{n-1} - a_{n-1})^2}{3a_{n-1}} \leq \frac{(c_{n-1} - a_{n-1})^2}{3a}$, $n = 1, 2, \ldots$

Proof: $c_n - a_n = \frac{a_{n-1} + b_{n-1} + c_{n-1}}{3} - \frac{3a_{n-1}b_{n-1}c_{n-1}}{b_{n-1}c_{n-1} + c_{n-1}a_{n-1} + a_{n-1}b_{n-1}}$

$$= \left[ a_{n-1}^2b_{n-1} + a_{n-1}b_{n-1}^2 + a_{n-1}^2c_{n-1} + \frac{a_{n-1}^2c_{n-1}}{3} + \frac{b_{n-1}^2c_{n-1}}{3} + \frac{b_{n-1}c_{n-1}^2}{3} - \frac{6a_{n-1}b_{n-1}c_{n-1}}{3} \right]$$

$$= \frac{a_{n-1}(b_{n-1} - c_{n-1})^2 + b_{n-1}(c_{n-1} - a_{n-1})^2 + c_{n-1}(a_{n-1} - b_{n-1})^2}{3(b_{n-1}c_{n-1} + c_{n-1}a_{n-1} + a_{n-1}b_{n-1})}$$

$$< \frac{(a_{n-1} + b_{n-1} + c_{n-1})(c_{n-1} - a_{n-1})^2}{3(b_{n-1}c_{n-1} + c_{n-1}a_{n-1} + a_{n-1}b_{n-1})}$$

$$< \frac{(a_{n-1} + b_{n-1} + c_{n-1})(c_{n-1} - a_{n-1})^2}{3(a_{n-1}c_{n-1} + a_{n-1}a_{n-1} + a_{n-1}b_{n-1})} < \frac{(c_{n-1} - a_{n-1})^2}{3a_{n-1}} \leq \frac{(c_{n-1} - a_{n-1})^2}{3a}.$$

Remark 3: Proposition 8 shows us that we have quadratic convergence. Therefore, it is possible to approximate AGHM $(a, b, c)$ fairly accurately in a few iterations.
Applications

In this section, we show that the original conjecture proposed by [1] is not true and provide a different application (that does not directly involve a surface integral) using the existence and properties of the AGHM.

**Conjecture:**

\[
\frac{1}{\text{AGH}(a,b,c)} = \frac{1}{2\pi} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \frac{d\phi d\theta}{\left[a^2 \cos^2 \theta \sin^2 \phi + b^2 \sin^2 \theta \sin^2 \phi + c^2 \cos^2 \phi\right]^{\frac{1}{2}}}.
\]

This conjecture was made in January 1989 by [1] and is not true because of the following:

- \(\text{AGM}(1,2)\) is approximately 1.4567550309.
- \(\text{AGHM}(1,1,2)\) is approximately 1.2632465593.

However, it is possible that the reciprocal of the AGHM is equal to some closed surface integral.

**RECENT APPLICATIONS**

Raissouli, Leazizi, Chergui [3] have used the existence and some of the properties of the AGHM in developing the Arithmetic-Geometric-Harmonic Mean of Three Positive Operators. However, it should be noted that the AGHM and its properties are not readily available in the literature.

**REFERENCES**