**On Some Suboptimal Estimators of the Singular Gauss Markov Model**

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**Abstract**

Using Rao’s generalized inverse, Srivastava and Khatri (1979) derive certain suboptimal estimators for the parameters of the singular Gauss Markov model. This paper presents a methodology for obtaining the context optimal solutions. The methodology is based on converting singular quadratic forms minimization theory to the nonsingular quadratic forms minimization theory.

**INTRODUCTION**

Given the minimum value problem (MVP)

\[(n-q), \]
\[\min y' Ay, \text{subject to } Dy = v, \quad (1)\]

Where the (column) vector \( y \) has \( n \) components, \( n \times nA \) is a symmetric positive definite, \( D \) is \( q \times n \) and of rank \( q<n \), \( v \) specified, Kabe [1] solves (1) by the following compact matrix transformation.

He writes the MVP (1) as

\[\min y' Ay, \text{subject to } (D'G)'y = (v'G)'t, \quad (2)\]

Where \( G_{(n-q)} \times n \) of rank, \( (n-q) \) and \( t \) are arbitrary.

The MVP (2) is the same as Min, Subject to

\[\left( A^{-1}D' A^{-1}G' \right) x = \left( v' \right) \quad (3)\]

The linear restrictions (3) yield the solution

\[x = \begin{pmatrix} \frac{D^4}{A} \end{pmatrix}^{-1} \begin{pmatrix} t \end{pmatrix} = \begin{pmatrix} \frac{D^4}{A} \end{pmatrix}^{-1} \begin{pmatrix} t \end{pmatrix} + \begin{pmatrix} \frac{D^4}{A} \end{pmatrix}^{-1} \begin{pmatrix} \frac{D^4}{A} \end{pmatrix}^{-1} \begin{pmatrix} 0 \end{pmatrix} \]

\[= \begin{pmatrix} \frac{A^{-1}D' A^{-1}G' \left( D^{-1}A' D^{-1}G' \right)^{-1} G \end{pmatrix} \begin{pmatrix} t \end{pmatrix} \]

Provided that \( DA^*G = 0 \). It follows that

\[y = A^*D' \left( D^*A^*D' \right)^{-1} v + C' (CAC')^{-1} t, DC' = 0, CA^*t = 0 \]

\[y = A^*D' \left( D^*A^*D' \right)^{-1} v + C' (CAC')^{-1} t, \quad (4)\]

If \( A \) is singular, the decomposition (7) does not hold if \( A^{-1} \) is replaced by \( A' \), where \( A' \) is Rao’s generalized inverse of \( A \).

To avoid the use if generalized inverses Kabe [2,3] writes (1) as

\[\min y' Ay, \text{subject to } Dy = v, CA^*t = 0 \]

If \( (A+D'D) \) is nonsingular, then use it for \( A \) in (8); however, if \( (A + D'D) \) is singular, then write (8) as

\[\min y' \left( A + C'C \right) y, \text{subject to } Dy = v, CA^*t = 0 \]

The matrix \( C \) can always be found such that \( (A+D') \) is nonsingular and either \( CA^*t = 0 \), or \( C(A+D') y = 0 \), and (9) is within the framework of (1).

We illustrate (9) by an example.

Example 1:

\[
A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \nu = 1 \tag{10}
\]

\[
T = A + CC^* = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad y_0 = 1. \tag{11}
\]

\[
C^*T^{-1}y_0 = (0, 0, 1)T^{-1}(0, 1, 0)' = 0 \tag{12}
\]

Thus when \( A \) is singular, \( y_0 \) is not unique, and hence the results must be changed by an optimality criterion of the type (12).

The MVP of type (8) frequently appears in the estimation theory of linear parametric functions of linear or quadratic parameters of the singular Gauss Markov theory. Section 2 studies the linear parametric estimation theory and mentions the problems of quadratic estimation theory. Section 3 points out some suboptimal solutions obtained for these types of estimation theory by Srivastava and Khatri [4].

**Singular gauss markov model problems**

Consider estimating \( \beta \) by the MVP

\[
\text{Min } (y - X\beta)'V'y - (y - X\beta) \tag{13}
\]

If \( V \) is nonsingular, then of course

\[
\hat{\beta} = (X'X)^{-1}X'y, \tag{14}
\]

is the BLUE, and

\[
(y - X\beta)'V^{-1}(y - X\beta) = (y - X\hat{\beta})'X'X(y - X\hat{\beta}). \tag{15}
\]

The decomposition (15) is not possible if \( V \) is singular and \( V^{-1} \) replaces \( V^{-1} \) in (15).

To solve (13), we follow (9) and write (13) to be subject to

\[
T(y - X\hat{\beta}) = 0, \tag{16}
\]

where \( n \times n' \) is of rank \( t \geq q, Xn \times q \) and of rank \( q < n \), and \( T \) is \((n - t) \times n \), which can always be found such that \( (V^{-1} + TT) \) is nonsingular, and \( E(T(y - X\beta)) = 0 \).

Example 2:

\[
V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T = t = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \tag{17}
\]

\[
G = (V^{-1} + TT) = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 4 \end{bmatrix} \tag{18}
\]

and the problem is within the framework of (9). It follows that

\[
2\hat{\mu} = y_1 + y_2 = 2y_3, E(t'(y - J\hat{\mu})) = 0. \tag{19}
\]

Next the variance component estimation problem finds quadratic minimum variance unbiased estimates of \( \sigma_1^2, \ldots, \sigma_r^2 \), for the model

\[
y = X\beta + e, e \sim N(0, \Omega), \Omega = \sum_{i=1}^{r} \sigma_i^2I, \tag{20}
\]

where \( \sigma_i \) are certain known idempotent specified matrices.

To solve the problem write \( y = y_1 + y_2, \)

\[
y_2 = (I - P)y_2, P = X(X'X)^{-1}X'V^{-1}X, E(y_2'Av_2) = trAM, \tag{21}
\]

\[
M = (I - P)\Omega(I - P), E(y_2) = 0, V = \sum_{s} = trAMAM, \tag{22}
\]

\[
V = V_1 + \ldots + V_s, \text{ and the MVP is Srivastava and Khatri [4].}
\]

Min \( trAMAM \), subject to \( AX = 0 \) or \( X'AX = 0 \), \( \text{and } A \) is symmetric idempotent.

This is a singular quadratic form MVP, and Srivastava and Khatri [4], (example 5.8.3) obtain a suboptimal solution to the problem (23).

The usual method of estimating variance or variance covariance components estimation by using n way classification ANOVA theory, at least, avoids the singular MVP’s of the type (23).

**Suboptimal solutions**

Srivastava and Khatri [4] section 5.4.2, equation (5.4.23)) estimate \( \beta \) for the model

\[
x = Z'\beta + e, e \sim N(0, \sigma^2W), Z, W, \text{ Singular}, \tag{24}
\]

And \( Z \) is \( q \times n \), \( \text{ rank } W \geq \text{ rank } Z \). Assuming \( Z \) to be of rank \( f \), they write

\[
Z = Z_1, Z_2, q \times r, Z_r \times n, \tag{25}
\]

\[
Z_1Z_1 = 0, \quad Z_2Z_1 = 0, Z_2Z_2 = \beta_1Z_2, \beta_1 = \Sigma, \tag{26}
\]

\[
\beta = Z_1(Z_1Z_2)'^{-1} \beta_1 + Z_2(Z_2Z_1)'^{-1} \beta_2, \tag{27}
\]

\[
Z' = Z_1Z_2, \beta = Z_1\beta_1, W_1 = W = Z'Z, \tag{28}
\]

and obtain the BLUE of \( \beta \) to be

\[
\tilde{\beta} = (ZW_1Z_2)'^{-1} ZW_1x, \tag{29}
\]

and they attribute the result (27) to Rao [5].

We proceed to show that (29) is certainly not the BLUE.

Example 3:

\[
y = \mu + \alpha_1 + e, E(\hat{\delta}) = 0, \tag{30}
\]

\[
W = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} \mu \\ \mu \end{bmatrix}, \tag{31}
\]

\[
W_1 = W + Z'Z = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & -2 \\ 2 & 2 & 4 \end{bmatrix} \tag{32}
\]
I : $2W_1^* = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $II : 2W_2^* = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{pmatrix}$

$Z = Z_1Z_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \neq 0$

$Z' = \beta = \beta_1 = 2\mu, \beta_2 = 2\mu, Z_1Z_2 = Z_2Z_1 = 0,$

and $\beta_3$ is not estimable. If $x = (1,1,2)^T$, then one choice of $W_1^*$ yields $\beta_1 = 1$, and second choice yields $\beta_1 = 2$, this is because $(ZW'_2Z_2')$ and $ZW'_2x$ are not invariant for all possible choices of $W_1^*$. The problem can be solved by using (1.6).

It follows that the result, Srivastava and Khatri [4] (equation (5.6.7)) for the estimation of the double linear parametric function of the GMANOVA model, is a suboptimal solution result.

Consider a one way classification model

$$y_j = \mu + \alpha_i + e_j, E(\delta_i) = 0, \text{var}$$

In $k$ blocks of $n$ plots each. Since $\text{cov}(\delta_i, e_j) = 0$,

$$E(\text{blockSS}) = (k-1)(n\sigma_0^2 + \sigma^2),$$

$$E(\text{errorsSS}) = k(n-1)\sigma^2,$$

and (32) can be solved to estimate $\sigma_0^2$ and $\sigma^2$.

**REFERENCES**

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