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Zygmund-Calderon Operators in the Weighted Variable Exponent Spaces

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Abstract

This article is dedicated to the Zygmund-Calderon operators in the variable exponent spaces $L^{p(\cdot)}$ with measurable function $p: R^n \to (1,\infty)$. We establish that if an operator $T(f)(x) = f * K$ with the kernel $\left|\partial_x^\alpha K(x)\right| \le C(\alpha)|x|^{-n-|\alpha|}, \quad |\alpha| \le 1$ satisfies the $\int_{R^n} \left|T(f)(x)\right|^{p(x)} \omega(x)dx \le c_2 \int_{R^n} \left|f(x)\right|^{p(x)} \omega(x)dx$ for all $f \in L^{p(\cdot)}(R^n)$, $p_S = ess \sup_{x\in R^n} p(x) < \infty$ then the weight $\omega = \frac{d\mu}{dx}$ belongs to $A_{p(\cdot)}$-class. The inverse is also true, thus, if the maximal operator is bounded in $L^{p(\cdot)}(R^n)$ and $\left|\partial_x^\alpha K(x)\right| \le C(\alpha)|x|^{-n-|\alpha|}, \quad |\alpha| \le 1$, then, the inequality $\int_{R^n} \left|T(f)(x)\right|^{p(x)} \omega(x)dx \le c_2 \int_{R^n} \left|f(x)\right|^{p(x)} \omega(x)dx$ holds for all $f \in L^{p(\cdot)}(R^n)$ and each $\omega \in A_{p(\cdot)}$.

INTRODUCTION

The variable Lévesque spaces were introduced in 1961 by I. Tsenov who considered the problem of approximation in the Lévesque spaces [1-18]. The variable Lévesque space with a measurable function $p: R^n \to (1,\infty)$ is the set of all measurable function on R^n the inequality $\omega = \frac{\cdot}{dx}$ holds for some positive values of the parameter λ. The norm of the variable Lévesque space $L^{p(\cdot)}$ is defined as an infimum

$$\|f\|_{L^{p(\cdot)}(R^n)} = \inf\left\{\lambda > 0 : \int\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty \le 1\right\}.$$

The classical Lévesque spaces L^p is a special case of $L^{p(\cdot)}$ when function $p: R^n \to (1,\infty)$ is constant.

The most prominent feature of $L^{p(\cdot)}$ is existence of an analog of the Holder inequality in the weaker form $\int|f(x)g(x)|dx \le \left(1 + \frac{1}{p_m} - \frac{1}{p_S}\right)\|f\|_{L^{p(\cdot)}}\|g\|_{L^{q(\cdot)}}$, Where $p_m = ess \inf p(x)$ and $p_S = ess \sup p(x)$.

There is an essential difference between the classical Lévesque spaces and the variable $L^{p(\cdot)}$, the necessary and sufficient requirement for the operator $\tau(z, f(x)) = f(x-z)$ of translation to be bounded on $L^{p(\cdot)}$ is that the function $p: R^n \to (1,\infty)$ be a constant. The corollary of this is that the Young lemma $\|f * g\|_{L^{p(\cdot)}} \le const\|f\|_{L^{p(\cdot)}}\|g\|_{L^1}$ holds for all $f \in L^{p(\cdot)}$ and all $g \in L^1$ if and only if exponential function $p(\cdot)$ is a constant.

Let M is a maximal operator then the inequality $\|\mathrm{M}(f)\|_{L^p(R^n,\mu)} \le \sqrt[p]{A}\|f\|_{L^p(R^n,\mu)}$ holds for all $f \in L^p(R^n,\mu)$, $d\mu(x) = \omega(x)dx$ if and only if the weight $\omega \in A_p$, the class A_p is characterized by inequality $\frac{1}{mes(B)}\int_B d\mu(x)\left(\frac{1}{mes(B)}\int_B \omega(x)^{\frac{1}{1-p}} dx\right)^{p-1} \le A$ holding fork balls *B*.

In 2008, L. Diening and P. Hasto [6,7] generalized classes A_p to the variable exponential Lebesgue spaces by demanding that the inequality

$$\sup_B \frac{1}{(mes(B))^{p(B)}}\|\omega 1_B\|_{L^1}\|\omega^{-1}1_B\|_{L^{\frac{q(\cdot)}{p(\cdot)}}} \le A$$

Holds for some constants, the minimum of these constants is the value of norm $\|\omega\|_{A_{p(\cdot)}}$.

Some pertinent to the subject literature reviews can be found in the L. Diening, P. Hasto works [6,7], without being complete, we present the list of some interesting research on the subject [1-25]. In this article, we consider a Zygmund-Calderon operator T_a [17] in the variable exponent spaces $L^{p(\cdot)}$ given in the form $T(f)(x) = \int_{R^n} K(x-y)f(y)dy$ for almost all $x \notin \operatorname{supp}(f)$, with a singular kernel K such that, for $|\alpha| \le 1$, the estimate $\left|\partial_x^\alpha K(x)\right| \le C(\alpha)|x|^{-n-|\alpha|}$ holds for all $x \in R^n$ with the exception of $x = 0$. We establish that assume $f \mapsto T(f)$ is a Zygmund-Calderon operator $T(f)(x) = f * K$ with the kernel under restriction $\left|\partial_x^\alpha K(x)\right| \le C(\alpha)|x|^{-n-|\alpha|}, \quad |\alpha| \le 1$, and the $L^{p(\cdot)}$-Condit

$$\int_{R^n} \left|T(f)(x)\right|^{p(x)} \omega(x)\,dx \le c_2 \int_{R^n} \left|f(x)\right|^{p(x)} \omega(x)\,dx$$

Holds for all $f \in L^{p(\cdot)}(R^n)$, $p_S = ess\sup_{x\in R^n} p(x) < \infty$, then the weight $\omega = \frac{d\mu}{dx}$ must belong to $A_{p(\cdot)}$-class. Also, we prove the inverse result, namely, presume the maximal operator is bounded in $L^{p(\cdot)}(R^n)$ and operator $f \mapsto T(f)$ defined as above, then, for each weight $\omega \in A_{p(\cdot)}$, the inequality

$\int_{R^n} \left|T(f)(x)\right|^{p(x)} \omega(x)\,dx \le c_2 \int_{R^n} \left|f(x)\right|^{p(x)} \omega(x)\,dx$ Holds for all $f \in L^{p(\cdot)}(R^n)$

Lebesgue Spaces With Variable Exponential

Let Ω be an open connected subspace of the R^n. Let $P(\Omega)$ be a subspace of $L^1(\Omega)$ such that $p(\cdot) \in P(\Omega)$, $p(\cdot): \Omega \to (1,\infty)$.

Definition 1: For given $p(\cdot) \in P(\Omega)$, we define the conjugate function $q(\cdot)$ by $q(x) = \frac{p(x)}{p(x)-1}$ for all $x \in \Omega$.

We denote $p_m(\tilde{\Omega}) = ess\inf_{x\in\tilde{\Omega}} p(x)$ and $p_S(\tilde{\Omega}) = ess\sup_{x\in\tilde{\Omega}} p(x)$ for fixed $\tilde{\Omega} \subset \Omega$.

Definition 2: For fixed $p(\cdot) \in P(\Omega)$, we define the functional ρ_p by

$$\rho_p(f) = \int_\Omega \left|f(x)\right|^{p(x)} dx \qquad (1)$$

For $f \in L^1(\Omega)$.

Straight forward considerations yield the following properties.

Properties 1: For fixed subset $\Omega \subset R^n$ and given $p(\cdot) \in P(\Omega)$, we have that

1) $\rho_p(f) \ge 0$ For all $f \in L^1(\Omega)$;

2) $\rho_p(f) = 0$ If and only if $f = 0$;

3) For all $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1$, the inequality

$$\rho_p(\alpha f + \beta g) \le \alpha \rho_p(f) + \beta \rho_p(g) \qquad (2)$$

Holds for all $f, g \in L^1(\Omega)$;

4) If $\left|f(x)\right| \ge \left|g(x)\right|$ almost everywhere and $\rho_p(f) < \infty$ then $\rho_p(f) \ge \rho_p(g)$ and if $\rho_p(f) > \rho_p(g)$ then $\left|f(x)\right| \ne \left|g(x)\right|$.

Definition 3: For fixed $p(\cdot) \in P(\Omega)$, we define a norm by

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \rho_p\left(\frac{f}{\lambda}\right) \le 1\right\} \qquad (3)$$

For measurable function *f*. The functional space $L^{p(\cdot)}(\Omega) = L^p(\Omega)$ consists of all measurable functions *f* such that $\|f\|_{L^{p(\cdot)}} < \infty$.

Similar to the classical Lebesgue spaces, for the $L^{p(\cdot)}(\Omega)$-spaces, we can formulate an analog of the Holder norm inequality.

Theorem 2: For fixed $p(\cdot) \in P(\Omega)$ and conjugation functions $q(\cdot)$, the inequality

$$\int_\Omega \left|f(x)g(x)\right| dx \le c_p \|f\|_{L^p} \|g\|_{L^q} \qquad (4)$$

With the constant $c_p = 1 + \frac{1}{p_m} - \frac{1}{p_S}$ for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$.

Proof: First, we show that $\rho_p\left(\frac{f}{\|f\|_{L^p}}\right) = 1$ holds for each $p(\cdot) \in P(\Omega)$ and all $f \in L^{p(\cdot)}(\Omega)$, $f \ne 0$. Indeed, for all λ, $0 < \lambda < \|f\|_{L^p}$, we have

$$\rho_p\left(\frac{f}{\lambda}\right) \le \left(\frac{\|f\|_{L^p}}{\lambda}\right)^{p_S} \rho_p\left(\frac{f}{\|f\|_{L^p}}\right),$$

Thus, there exists λ such that $\rho_p\left(\frac{f}{\lambda}\right) < 1$ but $\rho_p\left(\frac{f}{\|f\|_{L^p}}\right) \ge 1$.

Next, assuming $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, applying the Young inequality, we estimate

$$\int_\Omega \left|\frac{f(x)}{\|f\|_{L^p}} \frac{g(x)}{\|g\|_{L^q}}\right| dx \le$$

$$\le \int_\Omega \frac{1}{p(x)} \left|\frac{f(x)}{\|f\|_{L^p}}\right|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} \left|\frac{g(x)}{\|g\|_{L^q}}\right|^{q(x)} dx \le$$

$$\le \frac{1}{p_m} \rho_p\left(\frac{f}{\|f\|_{L^p}}\right) + \frac{1}{q_m} \rho_q\left(\frac{g}{\|g\|_{L^q}}\right) \le \left(1 + \frac{1}{p_m} - \frac{1}{p_S}\right),$$

Since $q_m = ess\inf q(x) = \frac{p_S}{p_S - 1}$. Thus, we obtain

$$\int_\Omega \left|f(x)g(x)\right| dx \le \left(1 + \frac{1}{p_m} - \frac{1}{p_S}\right) \|f\|_{L^p} \|g\|_{L^q}.$$

We formulate several fundamental properties of $L^{p(\cdot)}$-functions without proving them.

1. Let $f \in L^{p(\cdot)}$ then there exists a sum-presentation of f as $f_1 + f_2$ where $f_1 \in L^{p_S} \cap L^{p(\cdot)}$ and $f_2 \in L^{p_m} \cap L^{p(\cdot)}$.

2. The functional space C_0^∞ is dense in $L^{p(\cdot)}$, $p_S < \infty$.

3. Assume $\{f_k\} \subset L^{p(\cdot)}$ and $\lim_{k\to\infty} \|f_k - f\|_{L^{p(\cdot)}} = 0$, $f \in L^{p(\cdot)}$ then there exists a subsequence $\left\{f_{k(t)}\right\} \subset L^{p(\cdot)}$ that $\lim_{t\to\infty} f_{k(t)} = f$ for almost everywhere.

4. Each Cauchy sequence in $L^{p(\cdot)}$ converges in $L^{p(\cdot)}$.

5. Let $1 < p_m$ then the mapping $g \mapsto \Psi(g)$ define by

$$\Psi(g, f) = \int f(x) g(x)\,dx \qquad (5)$$

Is an isomorphism so that for each continuous linear functional $\Psi \in \left(L^{p(\cdot)}\right)^*$ there exists a uniquely defined element g of $L^{q(\cdot)}$ such that $\Psi = \Psi(g)$ and $\|g\|_{L^{q(\cdot)}} \approx \|\Psi\|$. The space $L^{p(\cdot)}$ is reflexive.

Weighted Classes A_p

First, we remind some general definitions from harmonic analysis and operator theory.

A classical maximal operator M on R^n, $n > 2$ is given by

$$\mathrm{M}(f)(x)=\sup_{r>0}\frac{1}{mes(B(r))}\int_{|y|<r}|f(x-y)|dy \quad (6)$$

For all arbitrary locally integral functions f and all balls $B(r)$ of radius $r>0$ in R^n, $n>2$.

Let measure μ be absolutely continuous with respect to Lebesgue measure. A functional class A_p consists of all weights $\omega(x)=\frac{d\mu(x)}{dx}$, which coincide with locally integral functions $\omega\in L^1_{loc}(R^n)$, such that the estimate

$$\frac{1}{mes(B)}\int_B d\mu(x)\left(\frac{1}{mes(B)}\int_B \omega(x)^{\frac{1}{1-p}}dx\right)^{p-1}\le A \quad (7)$$

Holds for all balls B, where $p+q=pq,\quad p\in(1,\infty)$. The A_p bound of the weight ω is a minimal constant for which (2) holds.

Applying the Holder inequality, we can prove the following lemma.

Lemma 1: For the weight ω to belong to A_p -class it is necessary and sufficient that the estimate

$$\frac{1}{mes(B)}\int_B|f(x)|dx\le c\left(\frac{1}{\mu(B)}\int_B|f(x)|^p d\mu(x)\right)^{\frac{1}{p}} \quad (8)$$

Holds for all $f\in L^1_{loc}(R^n)$ and all balls B, where $\omega(x)dx=d\mu(x)$.

Definition 4: The functional class BMO (bounded mean oscillation) consists of all locally integral functions f such that the inequality

$$\frac{1}{mes(B)}\int_B\left|f(x)-\frac{1}{mes(B)}\int_B f(x)dx\right|dx\le A \quad (9)$$

Holds for all balls B.

An important property of A_p - weights is given by the next theorem.

Theorem 3: Let $\omega\in A_p$ then the inequality

$$\int_{R^n}(\mathrm{M}(f)(x))^p\omega(x)dx\le A\int_{R^n}|f(x)|^p\omega(x)dx \quad (10)$$

Holds for all $f\in L^p(d\mu)$ and for each $p\in(1,\infty)$.

Now, we can generalize the definition of A_p - class to functional spaces $L^{p(\cdot)}(\Omega)$, $p(\cdot)\in P(\Omega)$.

Definition 5: For given $p(\cdot)\in P(\Omega)$, a weight $\omega(x)=\frac{d\mu(x)}{dx}$ belongs to the variable class $A_{p(\cdot)}$ if the inequality

$$\sup_B\frac{1}{(mes(B))^{p(B)}}\|\omega 1_B\|_{L^1}\|\omega^{-1}1_B\|_{L^{\frac{q(\cdot)}{p(\cdot)}}}\le A \quad (11)$$

Holds for all balls $B\subset clos(\Omega)$, and some constants A, where

$$p(B)=\left(\frac{1}{mes(B)}\int_B p(x)^{-1}dx\right)^{-1} \quad (12)$$

And $p(x)^{-1}+q(x)^{-1}=1$.

Theorem 4: Let $p(\cdot)\in P(\Omega)$ then a weight $\omega(x)=\frac{d\mu(x)}{dx}$ belongs to the variable $A_{p(\cdot)}$ - class if and only if the inequality

$$\left(\frac{1}{mes(B)}\int_B f(x)dx\right)^{p(B)}\le c_1(\mu(B))^{-1}\int_B f(x)^{p(x)}\omega(x)dx \quad (13)$$

Holds for all nonnegative $f\in L^{p(\cdot)}(d\mu)$ and all balls B.

Proof: Let $\omega\in A_{p(\cdot)}$ and applying the Holder inequality for variable Lebesgue spaces, we obtain

$$\left(\frac{1}{mes(B)}\int_B|f(x)|dx\right)^{p(B)}=$$

$$=\left(\frac{1}{mes(B)}\int_B f(x)\omega(x)^{\frac{1}{p(x)}}\omega(x)^{-\frac{1}{p(x)}}dx\right)^{p(B)}\le$$

$$\le c\frac{1}{mes(B)^{p(B)}}\left(\int_B(f(x))^{p(x)}\omega(x)dx\right)\|\omega^{-1}\|_{L^{\frac{q(\cdot)}{p(\cdot)}}},$$

Applying the definition of $A_{p(\cdot)}$-class, we deduce the first statement of the theorem.

Conversely, we take $f(x)=(\omega(x)+\varepsilon)^{-\frac{q(x)}{p(x)}}$ and have

$$\frac{1}{(mes(B))^{p(B)}}\|\omega 1_B\|_{L^1}\|(\omega+\varepsilon)^{-1}1_B\|_{L^{\frac{q(\cdot)}{p(\cdot)}}}\le c_1,$$

Take the limit as $\varepsilon\to 0$ and obtain that $\omega\in A_{p(\cdot)}$.

For the weighted variable Lebesgue space $L^{p(\cdot)}(R^n,\mu)$, one can prove an analog of Theorem 3 as follows.

Theorem (analog of theorem 3) 5: For fixed $p(\cdot)\in P(R^n)$ and weight $\omega\in A_{p(\cdot)}$, if maximal operator M is continuous on $L^{q(\cdot)}(d\mu)$ then the inequality

$$\int_{R^n}(\mathrm{M}(f)(x))^{p(x)}\omega(x)dx\le A\int_{R^n}|f(x)|^{p(x)}\omega(x)dx \quad (14)$$

Holds for all $f\in L^{p(\cdot)}(d\mu)$ and for some constants A

Pseudo-Differential Operators

The singular integral realization of the pseudo-differential operator T_a can be present as

$$T_a(f)(x)=\int_{R^n}k(x,y)f(x-y)dy \quad (15)$$

Or in classical form

$$T_a(f)(x) = \int_{R^n} K(x, y) f(y) dy \tag{16}$$

For almost all $x \notin \text{supp}(f)$, where $K(x, y) = k(x, x - y)$ and the Fourier transform

$$a(x, \xi) = \int_{R^n} \exp(-2\pi y \cdot \xi) k(x, y) dy. \tag{17}$$

We consider a convolution operator in the form

$$T(f)(x) = \int_{R^n} K(x - y) f(y) dy \tag{18}$$

For almost all $x \notin \text{supp}(f)$. The natural assumption on the integral kernel K is that there exists a smooth function $K(x)$, for all $x \in R^n$ except $x \neq 0$, such that the kernel agrees with $K(x)$ on elements of $C_0^\infty(R^n)$, which vanish on the neighborhood of $x = 0$, for all $|\alpha| \leq 1$ we assume

$$\left|\partial_x^\alpha K(x)\right| \leq C(\alpha)|x|^{-n-|\alpha|} \tag{19}$$

For all $x \in R^n$ except origin. This class of pseudo-differential operators satisfies the Zygmund-Calderon conditions.

Theorem 6: Let $p(\cdot) \in P(R^n)$ and let $f \mapsto T(f)$ be a convolution operator $T(f)(x) = f * K$ corresponding to kernel K such that $\left|\partial_x^\alpha K(x)\right| \leq C(\alpha)|x|^{-n-|\alpha|}$, $|\alpha| \leq 1$, $p_S = \underset{x \in R^n}{ess\sup}\, p(x) < \infty$. If the integral inequality

$$\int_{R^n} |T(f)(x)|^{p(x)} \omega(x) dx \leq c_2 \int_{R^n} |f(x)|^{p(x)} \omega(x) dx \tag{21}$$

Holds for all $f \in L^{p(\cdot)}(R^n)$ then the weight $\omega(x) = \frac{d\mu(x)}{dx}$ satisfies the inequality

$$\sup_B \frac{1}{(mes(B))^{p(B)}} \|\omega 1_B\|_{L^1} \|\omega^{-1} 1_B\|_{L^{\frac{q(\cdot)}{p(\cdot)}}} \leq A, \tag{22}$$

Namely $\omega \in A_{p(\cdot)}$.

Proof: Applying conditions on the kernel, we have

$$|K(x+z) - K(x)| \leq \breve{c}|x|^{-n}$$

For $|z| \leq \breve{c}|x|$. Assume $B(\tilde{x}, \rho)$ then we have that the inequality

$$|T(f)(x)| \geq \breve{c}_1 \frac{1}{(mes(B))^{p(B)}} |K(x - \tilde{x} - \rho\hat{x})| \int_{R^n} f(x)^{p(x)} dx$$

Holds for all $x \in B(\tilde{x} + \rho\hat{x}, \rho)$, the application of the integral inequality yields

$$mes(B(\tilde{x} + \rho\hat{x}, \rho)) \left(\frac{1}{mes(B)} \int_{R^n} f(x)^{p(x)} dx \right)^{p(B)} \leq c_2 \int_B f(x)^{p(x)} \omega(x) dx.$$

Taking $B(\tilde{x} + \rho\hat{x}, \rho)$ instead of $B(x, \rho)$, we obtain

$$mes(B) \left(\frac{1}{mes(B)} \int_B f(x) dx \right)^{p(B)} \leq c_2 \int_B f(x)^{p(x)} \omega(x) dx.$$

Theorem 7: Let $p(\cdot) \in P(R^n)$ such that $p_S = \underset{x \in R^n}{ess\sup}\, p(x) < \infty$, and let $f \mapsto T(f)$ be a convolution operator corresponding to kernel K under the assumption $\left|\partial_x^\alpha K(x)\right| \leq C(\alpha)|x|^{-n-|\alpha|}$, $|\alpha| \leq 1$ by $T(f)(x) = f * K$. Assume that the maximal operator is bounded in $L^{p(\cdot)}(R^n)$, then, for each weight $\omega \in A_{p(\cdot)}$, the inequality

$$\int_{R^n} |T(f)(x)|^{p(x)} \omega(x) dx \leq c_2 \int_{R^n} |f(x)|^{p(x)} \omega(x) dx \tag{23}$$

Holds for all $f \in L^{p(\cdot)}(R^n)$.

Proof: Let T_ε, $\varepsilon > 0$ be a truncated approximation with kernel $K_\varepsilon(x) = K(x) 1(\{|x| \geq \varepsilon\})$ so that

$$T_\varepsilon(f)(x) = \int_{R^n} K_\varepsilon(x - y) f(y) dy$$

And we define $T_*(f)(x) = \sup_\varepsilon |T_\varepsilon(f)(x)|$.

We show that the inequality

$$mes(\{x : T_*(f)(x) > \alpha, f(x) \leq \tilde{c}\alpha\}) \leq$$

$$\leq c_2 \tilde{c} (1 - \tilde{b})^{-1} mes(\{x : T_*(f)(x) > \tilde{b}\alpha\})$$

Holds for all $f \in C_0^\infty$ and for all $\tilde{b} < 1$, $\alpha > 0$ and $\tilde{c} > 0$.

Indeed, for fixed $\varepsilon > 0$ and for all $f \in C_0^\infty$, $T_\varepsilon(f)$ is a continuous function, therefore, there exists an open coverage Θ such that $T_*(f)(x) = \sup_\varepsilon |T_\varepsilon(f)(x)| > \tilde{b}\alpha$ is open, this open coverage Θ is decomposed into a disjoint union of Whitney cubes $\cup Q_i$.

Now, we decompose the function f into the sum $f_1 + f_2$ of two functions $f_1 = f 1_B$ and $f_1 = f 1_{R^n \setminus B}$. For $\tilde{b}_1 + \tilde{b}_2 = 1$ we obtain

$$\{x : T_*(f)(x) > \alpha\} \subset \{x : T_*(f_1)(x) > \tilde{b}_1\alpha\} \cup \{x : T_*(f_2)(x) > \tilde{b}_2\alpha\}.$$

Since for all $f \in L^1(R^n)$ there is an inequality

$$mes\{x : T_*(f)(x) > \alpha\} \leq \frac{c_2}{\alpha} \int_{R^n} |f(x)| dx,$$

We have

$$mes\{x \in Q_i : T_*(f_1)(x) > \tilde{b}_1\alpha\} \leq \frac{c_2}{\alpha\tilde{b}_1} \int_{R^n} |f_1(x)| dx$$

And

$$\int_{R^n} |f_1(x)| dx \leq \tilde{c} c_2 \alpha\, mes(Q_i),$$

Thus, we obtain

$$mes\{x : T_*(f_1)(x) > \tilde{b}_1\alpha\} \leq \tilde{c} \frac{c_2}{\tilde{b}_1} mes(Q_i).$$

Next, we must estimate the f_2-term. Applying our conditions, we calculate

$$\int_{|y-z| \geq s} \frac{|f(y)|}{|y-z|^{n+1}} dy = \int_{|y| \geq s} \frac{|f(z-y)|}{|y|^{n+1}} dy =$$

$$= \sum_{k=0,1,\ldots} \int_{2^i s \leq |y| \leq 2^{i+1} s} \frac{|f(z-y)|}{|y|^{n+1}} dy \leq 2\tilde{c}_1 f(z)$$

Since $|K_\varepsilon(\tilde{x} - y) - K_\varepsilon(x - y)| \leq \frac{c_2 s}{|y-z|^{n+1}}$ for $x \in Q_i$, $y \in R^n \setminus B$ and $R^n \setminus B \subset \{y : |y - z| \geq s\}$, the ball B has center at $\tilde{x}$. Therefore, we estimate

$$|T_\varepsilon(f_2)(\tilde{x}) - T_\varepsilon(f_2)(x)| \leq c_2 f(z)$$

For all $x \in Q_i$. Taking the supreme over all $\varepsilon > 0$, we have

$T_*(f_2)(x) \le T_*(f_2)(\tilde{x}) + c_2 f(z) \le \alpha\tilde{b} + c_2\tilde{c}\alpha$

For all $x \in Q_i$.

So, we choose $\tilde{b}_2 \ge \tilde{b} + c_2\tilde{c}$ and $\tilde{b}_1 + \tilde{b}_2 = 1$ then we have

$mes(\{x : T_*(f)(x) > \alpha, f(x) \le \tilde{c}\alpha\}) \le$

$\le c_2\tilde{c}\tilde{b}_1^{-1} mes(Q_i)$,

If $c_2\tilde{c}(1-\tilde{b})^{-1} \ge 2^{-1}$ then we choose new c_2 as $2c_2$ and obtain

$mes(\{x : T_*(f)(x) > \alpha, f(x) \le \tilde{c}\alpha\}) \le$

$\le c_2\tilde{c}(1-\tilde{b})^{-1} mes(Q_i)$.

Taking the sum over all cubes Q_i, we obtain

$mes(\{x : T_*(f)(x) > \alpha, f(x) \le \tilde{c}\alpha\}) \le$

$\le c_2\tilde{c}(1-\tilde{b})^{-1} mes(\{x : T_*(f)(x) > \tilde{b}\alpha\})$

For all $\alpha > 0$ and each $0 < \tilde{b} < 1$ and each $0 < \tilde{c}$.

Next, we show that the inequality

$\mu\{x : T_*(f)(x) > \alpha, f(x) \le \tilde{c}\alpha\} \le$

$\le \tilde{a}\mu\{x : T_*(f)(x) > \tilde{b}\alpha\}$

Holds for all $f \in C_0^\infty$ and for all $\tilde{b} < 1$, $\alpha > 0$ and some $\tilde{a} < 1$, $\tilde{c} > 0$, constant $\tilde{a}$ depends on the weight function.

Indeed, in previous consideration, we fix $\tilde{b} < 1$ and choose $\tilde{c}$ so that $c_2\tilde{c}(1-\tilde{b})^{-1}$ is small enough, then the inequality

$mes(\{x : T_*(f)(x) > \alpha, f(x) \le \tilde{c}\alpha\}) \le$

$\le \delta mes(Q_i)$.

Holds for some small enough positive δ and all cubes. Assuming $\tilde{a} = \delta < 1$, we summate over all cubes and obtain

$\mu(\{x : T_*(f)(x) > \alpha, f(x) \le \tilde{c}\alpha\}) \le$

$\le \delta\mu(Q)$.

We will need the following properties of the $p(\cdot)$-spaces. For constant $\tilde{a} < 1$, we choose $\tilde{b} < 1$ such that the inequality $\tilde{a} < \tilde{b}^{p_S}$ holds for all x. Let f and g be a nonnegative function such that inequality

$\mu(\{x : f(x) > \alpha, g(x) \le \tilde{c}\alpha\}) \le$

$\le \tilde{a}\mu(\{x : f(x) > \tilde{b}\alpha\})$

Holds for all $\alpha > 0$. Then, the inequality

$$\int_{R^n} |f(x)|^{p(x)} d\mu(x) \le c_3 \int_{R^n} |g(x)|^{p(x)} d\mu(x)$$

Holds with some constants c_3 and under the condition $\tilde{a} < \tilde{b}^{p_S}$ and $f \in L^{p(\cdot)}$. Proving of this statement is similar to standard one.

This proves our statement for $f \in C_0^\infty$ since $|T_*(f)(x)| \le C(1+|x|)^{-n}$ holds for all $f \in C_0^\infty$. The extension to the whole $L^{p(\cdot)}(R^n, \mu)$ follows from the standard argument that each element of $L^{p(\cdot)}(R^n, \mu)$ can be approximated by ele elements of $C_0^\infty(R^n)$.

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